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Integrable generalized spin ladder models based on the SU(1|3) and SU(3|1) algebras

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We present two integrable spin ladder models which possess a general free parameter besides the rung coupling J. The models are exactly solvable by means of the Bethe ansatz method and we present the Bethe ansatz equations. We analyze the elementary excitations of the models which reveal the existence of a gap for both models that depends on the free parameter. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1627973]

I. INTRODUCTION

Spin ladder systems continue to attract attention motivated by experimental realizations in quasi-one-dimensional systems.[1] These materials display novel features and with the continued development of new systems, there has been an impressive amount of progress in the theoretical understanding of such systems. However, a greater flexibility through the introduction of tunable free parameters within the well established mathematical frameworks would be of considerable advantage and forms the main aim of the present work.

It has been shown that ladder systems are reasonably well approximated by Heisenberg interactions, which involve bilinear exchanges.[2] While these models are not exactly solvable, several more general systems have been proposed in which solvability is guaranteed through the use of an extension of the symmetry algebra.[3–6] There has also been the introduction of systems involving interactions beyond nearest neighbor exchanges which demonstrate remarkably interesting behavior and also prove to be exactly solvable. For example, the general 2-leg spin ladder system with biquadratic interactions.[7,8] The physical importance of these types of interactions has been addressed in Ref. 9.

Subsequently other generalized integrable spin ladders have been proposed.[10–14] As is well known, integrability facilitates the use of long established techniques in order to determine the physical properties of such systems. However, in these cases, no free parameters other than the rung coupling are present due to the strict conditions of integrability. It is clear that this is a topic that warrants further investigation, since the availability of tunable parameters yields a richer phase structure.

In this article, we present two new integrable generalized spin ladders containing an extra parameter, based on the Lie superalgebras symmetries of SU(1|3) and SU(3|1). The free parameter arises in the models as a special choice of the multiparametric versions.[15] In Ref. 16, we note the

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study of a family of spin ladder Hamiltonians which also have free parameters, although in this case the construction has a different mathematical origin.

The models are integrable in the sense that they contain an infinite number of conservation laws and can be derived from a solution of the Yang–Baxter equation. This property is also of physical importance as it provides a means to improve our understanding of general correlated systems (see, for example, Ref. 17).

The Hamiltonians involve the usual bilinear Heisenberg interaction terms as well as biquadratic exchanges to ensure integrability. These four-spin terms represent interchain coupling and inter-rung coupling and the physical justification for these types of interactions has been supported by experimental results. A discussion may be found in Refs. 8, 9, and 18. We present the Bethe ansatz solution of these Hamiltonians from which the physical properties of the systems may be obtained.

An important characteristic of ladder systems, both from a theoretical and experimental point of view, is the quantum phase transition between gapped and gapless phases. The spin gap is vital for superconductivity to occur under doping, whilst from a mathematical perspective, the size of the gap is dependent on the relative strength of the rung interaction parameter. We address this issue as we analyze the ground state and first excited states of the models. Interestingly, we are able to show that for both systems a gap persists in the spectrum of the elementary excitations and indeed the gap depends on the extra parameter.

II. GENERALIZED SU(1|3) SPIN LADDER MODEL

We begin by introducing the first generalized spin ladder model, for which the explicit global Hamiltonian is of the form

$$H^{(1)}=\sum_{j=1}^{N}\left[\, h_{j,j+1}+\tfrac{1}{2}J(\boldsymbol{\sigma}_j\,.\,\boldsymbol{\tau}_j-1)\,\right], \tag{1}$$

and the local Hamiltonians are given by

$$\begin{aligned} h_{j,j+1}=&\tfrac{1}{4}(1+\sigma_j^z\sigma_{j+1}^z)(1+\tau_j^z\tau_{j+1}^z)+(\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+)(\tau_j^+\tau_{j+1}^-+\tau_j^-\tau_{j+1}^+)\\ &+\tfrac{1}{2}(1+\sigma_j^z\sigma_{j+1}^z)(t^{-1}\,\tau_j^+\tau_{j+1}^-+t\,\tau_j^-\tau_{j+1}^+)+\tfrac{1}{2}(t^{-1}\,\sigma_j^+\sigma_{j+1}^-+t\,\sigma_j^-\sigma_{j+1}^+)(1+\tau_j^z\tau_{j+1}^z)\\ &-\tfrac{1}{8}(1+\sigma_j^z)(1+\sigma_{j+1}^z)(1+\tau_j^z)(1+\tau_{j+1}^z). \end{aligned}$$

The parameters $\boldsymbol{\sigma}_j$ and $\boldsymbol{\tau}_j$ represent Pauli matrices acting on site j of the upper and lower legs, respectively, J is the strength of the rung coupling that can take arbitrary real values, and t is a free parameter. The number of rungs is denoted by N and periodic boundary conditions are imposed.

The integrability of this model is assured by the Quantum Inverse Scattering Method[19] and by the fact that it can be mapped to the Hamiltonian given in Eq. (2) below. This Hamiltonian can be derived from an R-matrix obeying the Yang–Baxter algebra[20] for $J=0$, while for $J\neq 0$, the rung interactions take the form of a chemical potential term. We find that

$$\hat{H}^{(1)}=\sum_{j=1}^{N}\,[\hat{h}_{j,j+1}-2J\,X_j^{00}], \tag{2}$$

where

$$\begin{aligned} \hat{h}_{j,j+1}=&\sum_{\alpha=0}^{3}X_j^{\alpha\alpha}X_{j+1}^{\alpha\alpha}+X_j^{20}X_{j+1}^{02}+X_j^{02}X_{j+1}^{20}+X_j^{13}X_{j+1}^{31}+X_j^{31}X_{j+1}^{13}+t(X_j^{10}X_{j+1}^{01}+X_j^{12}X_{j+1}^{21}\\ &+X_j^{03}X_{j+1}^{30}+X_j^{23}X_{j+1}^{32})+t^{-1}(X_j^{01}X_{j+1}^{10}+X_j^{21}X_{j+1}^{12}+X_j^{30}X_{j+1}^{03}+X_j^{32}X_{j+1}^{23})-2X_j^{00}X_{j+1}^{00}. \end{aligned}$$

In the above, $X_j^{\alpha\beta}=|\alpha_j\rangle\langle\beta_j|$ are the Hubbard operators with $|\alpha_j\rangle$ being the orthogonalized eigenstates of the local operator $(\boldsymbol{\sigma}_j\,.\,\boldsymbol{\tau}_j)$. The local Hamiltonians $h_{j,j+1}$ and $\hat{h}_{j,j+1}$ are related through the following basis transformation:

$$
\begin{aligned}
&|\uparrow,\uparrow\rangle \to |\uparrow,\uparrow\rangle,\\
&|\uparrow,\downarrow\rangle \to 1/\sqrt{2}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle),\\
&|\downarrow,\uparrow\rangle \to 1/\sqrt{2}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle),\\
&|\downarrow,\downarrow\rangle. \to |\downarrow,\downarrow\rangle.
\end{aligned}
\tag{3}
$$

The R-matrix we use is a special case of a more general multiparametric version. (For a general construction of multiparametric models, see Ref. 15.) For the purposes of the present work, it is necessary to only retain one parameter. The R-matrix is as follows:

$$
R(x)=\left(\begin{array}{cccc|cccc|cccc|cccc}
w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & t^{-1}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0\\
\hline
0 & c & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0\\
\hline
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0\\
\hline
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-1}b & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{array}\right),
\tag{4}
$$

with

$$
a=x+1,\ \ b=x,\ \ c=1,\ \text{and}\ \ w=1-x,
$$

and obeys the Yang–Baxter algebra,[20]

$$
R_{12}(x-y)R_{13}(x)R_{23}(y)=R_{23}(y)R_{13}(x)R_{12}(x-y).
\tag{5}
$$

The Hamiltonian originates from this solution (2) for $J=0$ by the standard procedure,[19]

$$
\hat{h}_{j,j+1}=P\frac{d}{dx}R(x)|_{x=0},
$$

where P is the permutation operator.

The model is exactly solvable by the Bethe ansatz method[21] and the resulting Bethe ansatz equations (BAE) are given by the expressions

$$-(-1)^{M_1} t^{(N-2M_3)}\left(\frac{\lambda_l - i/2}{\lambda_l + i/2}\right)^N = \prod_{j=1}^{M_2} \frac{\lambda_l - \mu_j - i/2}{\lambda_l - \mu_j + i/2},$$

$$t^{(N-2M_3)}\prod_{j\neq l}^{M_2} \frac{\mu_l - \mu_j - i}{\mu_l - \mu_j + i} = \prod_{i=1}^{M_1} \frac{\mu_l - \lambda_i - i/2}{\mu_l - \lambda_i + i/2}\prod_{k=1}^{M_3} \frac{\mu_l - \nu_k - i/2}{\mu_l - \nu_i + i/2}, \tag{6}$$

$$t^{(N-2M_1+2M_2)}\prod_{k\neq l}^{M_3} \frac{\nu_l - \nu_k - i}{\nu_l - \nu_k + i} = \prod_{j=1}^{M_2} \frac{\nu_l - \mu_j - i/2}{\nu_l - \mu_j + i/2}.$$

The corresponding energy eigenvalues of the Hamiltonian (2) are

$$E = \sum_{j=1}^{M_1}\left(\frac{1}{\lambda_j^2 + 1/4} + 2J\right) - (1+2J)N, \tag{7}$$

where λ_j are solutions of the BAE (6).

From the Bethe ansatz solution, we can determine the behavior of the ground state and elementary excitations of the system. The reference state becomes the ground state when the relation $J > -1 + \frac{1}{2}(t+t^{-1})$ is satisfied. For N sites, the ground state energy is $E_0 = -(1+2J)N$, which in terms of the Bethe ansatz calculations, corresponds to the reference state characterized by $M_1 = M_2 = M_3 = 0$.

To describe an elementary excitation, we chose $M_2 = M_3 = 0$ and $M_1 = 1$ in the BAE which, from Eq. (7), yields an energy expression of the form

$$E_1 = \frac{1}{\lambda^2 + 1/4} + 2J - (1+2J)N, \tag{8}$$

where $\lambda = (i/2)((t+1)/(t-1))$. It is apparent that there is a gap of

$$\Delta = 2(J + 1 - \tfrac{1}{2}(t+t^{-1})). \tag{9}$$

By solving $\Delta = 0$ for J we find the critical value $J^c = -1 + \frac{1}{2}(t+t^{-1})$, indicating the critical line at which the quantum phase transition from the dimerized phase to the gapless phase occurs. This phase transition assumes a simpler form after a suitable reparametrization. We introduce a new parameter K given by $K = (t+t^{-1})/2$. In Fig. 1 the phase diagram is represented in terms of K and J. The phase boundary is now a straight line given by $J = -1 + K$.

In the limit $t=1$, this solution corresponds to $\lambda \to \infty$ indicating that a gap of $\Delta = 2J$ persists. We note that this is agrees with the suggested numerical and experimental results of spin ladder systems.[1]

The model also exhibits elementary bound state excitations. For example by choosing $\{M_1 = M_2 = 1, M_3 = 0\}$ there is solution of the Bethe ansatz equations given by

$$\lambda_1 = 0, \quad \mu_1 = \frac{i}{2}\,\frac{t^2+1}{t^2-1}$$

which describes an excited state of energy $E = 4 - N + 2J(1-N)$ and total spin zero.

III. GENERALIZED SU(3|1) SPIN LADDER MODEL

We move on to introduce the second integrable spin ladder model which also contains the free parameter. The global Hamiltonian reads

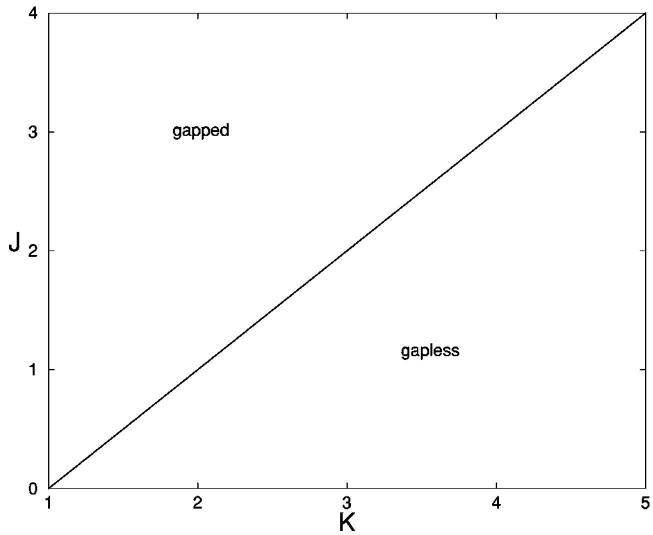

FIG. 1. Rung coupling J versus reparametrization parameter K. This graph represents the phase diagram between gapped and gapless phases. In this parametrization, the phase boundary is a straight line.

$$\mathcal{H}^{(2)}=\sum_{j=1}^{N}\left[k_{j,j+1}+\tfrac{1}{2}J(\boldsymbol{\sigma}_j\,.\,\boldsymbol{\tau}_j-1)\right],\tag{10}$$

where

$$\begin{aligned}k_{j,j+1}=&\tfrac{1}{4}(1+\sigma_j^z\sigma_{j+1}^z)(1+\tau_j^z\tau_{j+1}^z)+(\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+)(\tau_j^+\tau_{j+1}^-+\tau_j^-\tau_{j+1}^+)\\&+\tfrac{1}{2}(1+\sigma_j^z\sigma_{j+1}^z)(t^{-1}\tau_j^+\tau_{j+1}^-+t\,\tau_j^-\tau_{j+1}^+)+\tfrac{1}{2}(t^{-1}\sigma_j^+\sigma_{j+1}^-+t\,\sigma_j^-\sigma_{j+1}^+)(1+\tau_j^z\tau_{j+1}^z)\\&-\tfrac{1}{8}(1-\sigma_j^z)(1-\sigma_{j+1}^z)(1-\tau_j^z)(1-\tau_{j+1}^z).\end{aligned}$$

The exact solvability of the above Hamiltonian, as for the previous case, lies in the fact that it too can be mapped to a Hamiltonian given below by Eq. (11), via the transformation (3). Once again this Hamiltonian is derived from an R-matrix solution of the Yang–Baxter algebra for $J=0$, while for $J\neq 0$ the rung interactions take the form of a chemical potential term. The Hamiltonian has the form

$$\hat{\mathcal{H}}^{(2)}=\sum_{j=1}^{N}\left[\hat{k}_{j,j+1}-2JX_j^{00}\right],\tag{11}$$

where

$$\begin{aligned}\hat{k}_{j,j+1}=&\sum_{\alpha=0}^{3}X_j^{\alpha\alpha}X_{j+1}^{\alpha\alpha}+X_j^{20}X_{j+1}^{02}+X_j^{02}X_{j+1}^{20}+X_j^{13}X_{j+1}^{31}+X_j^{31}X_{j+1}^{13}+t(X_j^{10}X_{j+1}^{01}+X_j^{12}X_{j+1}^{21}\\&+X_j^{03}X_{j+1}^{30}+X_j^{23}X_{j+1}^{32})+t^{-1}(X_j^{01}X_{j+1}^{10}+X_j^{21}X_{j+1}^{12}+X_j^{30}X_{j+1}^{03}+X_j^{32}X_{j+1}^{23})-2X_j^{33}X_{j+1}^{33}.\end{aligned}$$

For $J=0$, the model is derived, in a similar manner as for the above case, from a multiparametric R-matrix for which only one parameter is relevant for the present discussion. The R-matrix is given by

$$
R(x)=\left(\begin{array}{cccc|cccc|cccc|cccc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t^{-1}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\
\hline
0 & c & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\
\hline
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0 \\
\hline
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-1}b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w
\end{array}\right), \tag{12}
$$

with

$$a=x+1,\ \ b=x,\ \ c=1\ \text{ and } w=-x+1,$$

and satisfies the Yang–Baxter algebra (5). Utilizing the Bethe ansatz method this model can be solved and the resulting BAE are

$$
\begin{aligned}
t^{(N-2M_3)}\left(\frac{\lambda_l-i/2}{\lambda_l+i/2}\right)^N &= \prod_{l\neq i}^{M_1}\frac{\lambda_l-\lambda_i-i}{\lambda_l-\lambda_i+i}\prod_{j=1}^{M_2}\frac{\lambda_l-\mu_j+i/2}{\lambda_l-\mu_j-i/2},\\
t^{(N-2M_3)}\prod_{j\neq l}^{M_2}\frac{\mu_l-\mu_j-i}{\mu_l-\mu_j+i} &= \prod_{i=1}^{M_1}\frac{\mu_l-\lambda_i-i/2}{\mu_l-\lambda_i+i/2}\prod_{k=1}^{M_3}\frac{\mu_l-\nu_k-i/2}{\mu_l-\nu_i+i/2},\\
-(-1)^{M_3}t^{-(N-2M_1+2M_2)} &= \prod_{j=1}^{M_2}\frac{\nu_l-\mu_j-i/2}{\nu_l-\mu_j+i/2}.
\end{aligned} \tag{13}
$$

The eigenenergies of the Hamiltonian (10) are given by

$$\mathcal{E}=-\sum_{j=1}^{M_1}\left(\frac{1}{\lambda_j^2+1/4}-2J\right)+(1-2J)N, \tag{14}$$

where λ_j are solutions of the BAE (13).

For N sites, the ground state is given by a product of rung singlets when $J>1+\frac{1}{2}(t+t^{-1})$ and the energy is $\mathcal{E}_0=(1-2J)N$. This is in fact the reference state used in the Bethe ansatz calculations and corresponds to the case $M_1=M_2=M_3=0$ of the BAE (13). To describe an elementary spin-1 excitation, we choose $M_1=1$ and $M_2=M_3=0$ in the BAE which gives the minimal excited state energy,

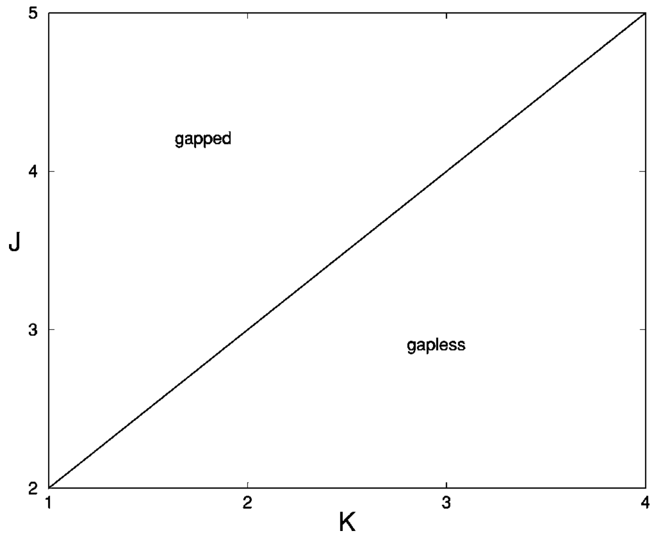

FIG. 2. Rung coupling J versus reparametrization parameter K. This graph shows a reparametrization of the curve $J = 1+(t+t^{-1})/2$ in terms of $K=(t+t^{-1})/2$. In this parametrization, the phase boundary is a straight line.

$$\mathcal{E}_1 = -\frac{1}{\lambda^2+1/4}+2J+(1-2J)N, \tag{15}$$

where $\lambda=(i/2)((t-1)/(t+1))$. The energy gap can easily be calculated and is found to be

$$\Delta = 2(J-1-\tfrac{1}{2}(t+t^{-1})). \tag{16}$$

The value $J^c = 1+\frac{1}{2}(t+t^{-1})$ indicates the critical line at which the transition from dimerized phase to the gapless phase occurs. This graphic is presented in Fig. 2 in terms of the reparametrization variable $K=(t+t^{-1})/2$.

The model also exhibits elementary bound state excitations. For example, by choosing $\{M_1 = M_2 = 1, M_3 = 0\}$ there is solution of the Bethe ansatz equations given by

$$\lambda_1 = 0, \quad \mu_1 = \frac{i}{2}\frac{t^2-1}{t^2+1},$$

which describes an excited state of energy $\mathcal{E} = -4+N+2J(1-N)$ and total spin zero.

IV. CONCLUSION

We have presented two new spin ladder models derived as special cases of multiparametric versions of Lie superalgebra SU(3|1) and SU(1|3) invariant solutions of the Yang–Baxter equation, maintaining one free parameter besides the rung coupling J. Upon investigation of the solutions of the BAEs to determine ground state and elementary excitations, we have shown that both models exhibit a gap that depends on the extra parameter. Our results are suggestive that such multiparametric extensions will, in general, have an influence on the physical characteristics of these models, and in particular the critical value of the rung coupling.

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